Higher homotopy groups of spheres

Mendel Keller

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All of the material of this paper follows Chapter 4 in Hatcher.

1 Introductory definitions

We can view the notion of a fundamental group as a functor $\pi_1 : \texttt{Top} \to \texttt{Grp}$, given by pointed maps $f : (S^1, 1) \to (X, x_0)$ up to homotopy. Similarly, we can define higher homotopy groups using a functor $\pi_n : \texttt{Top} \to \texttt{Ab}$ given by pointed maps $f : (S^n, 1) \to (X, x_0)$ up to homotopy. Another useful way of imagining this is as maps $f : (I^n, \partial I^n) \to (X, x_0)$ which maps the unit hypercube into X in such that the boundary is sent to a point. As is the case with π_1, the higher homotopy groups are base-point invariant, so we will often omit reference to a base-point for simplicity. We can immediately justify that these groups are commutative using the following sequence of homotopies.

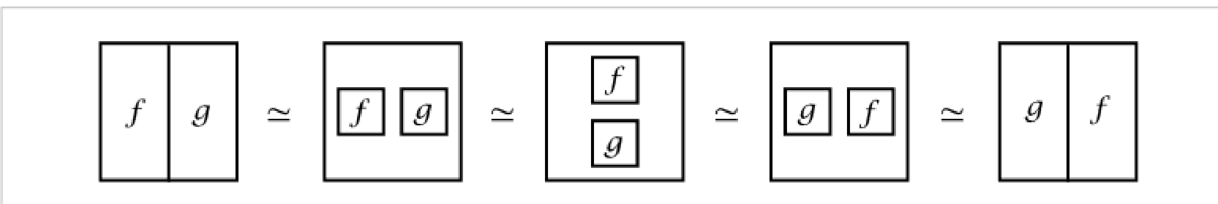

Figure 1: Shuffling a composite map using homotopy. (Hatcher pg 340)

This figure should be interpreted as, given two maps f, g of S^n into X, we compose them to get the first figure. Then shrink the domain of f and g, increasing the area of I^n that is mapped to x_0, we can then shuffle f and g around, and then expand their area again, in a continuous fashion, so that $f + g \cong g + f$.

We will only be considering the higher homotopy groups of spheres. It turns out that these are in general somewhat complex, and difficult to compute. There are many nontrivial homotopy groups that are created with elaborate constructions. But we will focus on the relatively well-behaved cases.

2 Homotopy groups of a circle

Theorem 1. *For $n > 1$, the homotopy group $\pi_n(S^1)$ is trivial.*

The idea here is we will be using the cover $\mathbb{R}$ of S^1. It turns out that we can factor maps from S^n to S^1 through $\mathbb{R}$. This will show that all these maps are trivial i.e. homotopic to a constant map, because any map into $\mathbb{R}$ is homotopic to a constant map.

Proof. Recall the lifting criterion. It says that given a covering space $p : \widetilde{X} \to X$, and a map $f : Y \to X$ into the space. We can create a lift $\tilde{f}$ of f to the covering space $\widetilde{X}$ iff the maps of groups into $\pi_1(X)$ induced by f and p satisfy $f_*(\pi_1(Y)) \subset p_*(\pi_1(\widetilde{X}))$.

Now, given any map $f : (S^n, 1) \to (S^1, 1)$, we can apply the lifting criterion to the cover $p : \mathbb{R} \to S^1$. It applies in this case because the fundamental group $\pi_1(S^n)$ is trivial. Thus the induced group homomorphism $f_* : \pi_1(S^n) \to \pi_1(S^1)$ must be simply mapping the identity to the identity. So we get that the image of the induced group map $f_*(\pi_1(S^n))$ will be just the identity, which is contained in any subgroup. In particular, it is contained in $p_*(\pi_1(\mathbb{R}))$. Thus we have inclusion of subgroups, and this satisfies the lifting criterion.

This now allows us to create a lift $\tilde{f} : S^n \to \mathbb{R}$, such that $f = p \circ \tilde{f}$. But $\pi_n(\mathbb{R}) = 0$, since we can simply choose a linear homotopy sending everything directly to the basepoint. Explicitly we can take the homotopy $f_t(x) = (1-t)x$. So now we know that every element of $\pi_n(S^1)$ can be factored through the identity, thus is the identity. In other words the image of the induced homomorphism $p_* : \pi_n(\mathbb{R}) \to \pi_n(S^1)$ is the trivial group. But every element of $\pi_n(S^1)$ factors through p, so this gives that $\pi_n(S^1) = 0$, as desired. ■

Remark 2. This theorem generalizes to show that $\pi_n(\widetilde{X}) \cong \pi_n(X)$. The injectivity is given by the homotopy lifting property, and the surjectivity is given by the lifting criterion and the fact that S^n is simply connected. We can then use this for example to show that the higher homotopy groups of the torus $\pi_n(S^1 \times S^1)$ are trivial, since it has cover $\mathbb{R}^2$.

3 Other trivial examples of trivial groups

Another set of trivial homotopy groups is $\pi_n(S^m)$ for $m > n$. We already know this to be the case for $n = 1$ i.e. we know that $\pi_1(S^n) = 0$ for $n > 1$. The intuition in that case is that S^1 isn't "big enough" to wrap around S^n. A similar intuition suggests that an analogous fact would be the case in higher dimensions as well. In order to actually prove this, we use the cellular approximation theorem.

Recall that cellular approximation said that if we have a map between cell complexes $f : X \to Y$, we can homotopy the map to make it cellular. Recall again that a cellular map is defined as the image of X^n (the n-skeleton) is contained in Y^n, symbolically $f[X^n] \subset Y^n$. Essentially, this is a continuous map which is compatible with the cell structures of X and Y. We can also choose this homotopy such that it is constant on the parts that are already cellular.

In particular, since we can build S^n as a point (which we then will choose as our basepoint) with an attached $n - cell$. As you will recall, we can imagine this as $\mathbb{E}^n$ with the boundary identified. Given a map $f : S^n \to S^m$ i.e. a representative of an element of $\pi_n(S^m)$, we can then use cellular approximation on it. When we use cellular approximation on a map $S^n \to S^m$ with $m > n$, we find that the image of S^n is homotopic (via base-point preserving homotopy) to a point. This is true because the n-skeleton of S^m is just a point, because we just keep a single point until we get to the m cells, but $m > n$, so we still only have a point as our n-skeleton of S^m. Thus every element of $\pi_n(S^m)$ is homotopic to a constant map, and we have shown that $\pi_n(S^m) = 0$ for $m > n$.

These are not the only higher homotopy groups of sphere that are trivial, in fact there are other infinite families of trivial higher homotopy groups of spheres. But these are the only simple examples that can be done without a whole bunch of fancy computation, as well as the only ones which can be relatively easily appreciated intuitively. The intuition in this case being that S^n isn't big enough to truly wrap around S^m, much in the same way that $\pi_1(S^n)$ is trivial for $n > 1$.

4 Stability

We have a third unsurprising fact about π_n, and that is that $\pi_n(S^n) = \mathbb{Z}$. However, this fact will be significantly more difficult to demonstrate than the previous two, and will require the introduction of a whole bunch more theory. We will delve into relative homotopy, and for the last bit use some fancier nontrivial maps. But essentially for this proof we will be simply using induction. We won't actually complete this proof here, but will save the last bit until later. For now we will use stability to show that $\pi_n(S^n) \cong \pi_{n+1}(S^{n+1})$ for $n > 1$. And we will later show that $\pi_1(S^1) \cong \pi_2(S^2)$ using a different method, which will then complete the proof and show that $\pi_n(S^n) = \mathbb{Z}$ for all n.

4.1 Relative homotopy

The same way that homotopy is given by maps $f : (S^n, 1) \to (X, x_0)$, we can define a related concept of relative homotopy. Instead of maps of pairs, this will be defined as maps of triplets. The third thing being introduced will be a subspace $A \subset X$ with $x_0 \in A$. The definition is a bit more technical than regular homotopy, as there are some subtleties lying within. A good picture to keep in mind is that if we take a map f inside the relative homotopy, and then consider the natural inclusion into X/A obtained by collapsing A to a point, this will give us a map $\overline{f}$ in the regular homotopy. That is essentially the best way to imagine what's going on, except that being precise in this case requires some subtlety and care.

Having said that, we now define relative homotopy. Given a subspace $A \subset X$ and a basepoint $x_0 \in A$ the relative homotopy is given by maps $f : (\mathbb{E}^n, S^{n-1}, 1) \to (X, A, x_0)$ modulo homotopy. Another more formal perspective on this is given by defining $J^{n-1} \subset I^n$. We first define $I^{n-1} \subset I^n$ as the face of I^n, the unit hypercube in dimension n, given by all points with n-th coordinate 0. Then define $J^{n-1} = (\partial I^n - I^{n-1}) \cup \partial I^{n-1}$. In words, this is the boundary of I^n missing the interior of I^{n-1}. We then have the relative homotopy given by equivalence classes of maps $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ modulo homotopy.

Note that, for example, any map $(S^n, 1) \to (X, x_0)$ also defines a map in the relative homotopy. So we have these, plus also some more things. For those other things we allow some of the boundary of I^n to leave x_0 and spill out into A. This might be imagined as a sphere that isn't completely sealed off, but rather has a hole, entirely contained in A. In this picture, you might wonder why this map isn't homotopic to a point as it would normally, since it is a disk without the boundary collapsed, and thus retractable. And the reason this isn't the case is because the S^{n-1} sized hole in S^n cannot leave A. Thus you might also imagine relative homotopy as a disk mapping into a space in a way that its boundary is trapped in some subspace. We can also see that, as noted above, if we collapsed A to a point these relative homotopy maps would turn into homotopy maps. Either way, we have defined it as a map of triplets, and so we can now work with it mathematically.

4.2 Long exact sequence of relative homotopy

Before we deal with relative homotopy more, we note that the zero elements of $\pi_n(X, A, x_0)$ are exactly those maps homotopic to a map which sends all of $\mathbb{E}^n$ into A. Essentially this works because we no longer face the issue of not being able to retract $\mathbb{E}^n$, or essentially by the fact that $\mathbb{E}^n$ is retractable. The other direction holds as well, since, given a homotopy $\mathbb{E}^n \times I \to X$ between a map f and the constant map, we can take sub"disks" of $\mathbb{E}^n \times I$ which range from $\mathbb{E}^n \times \{0\}$ to $\mathbb{E}^n \times \{1\} \cup S^{n-1} \times I$, i.e. always including the boundary of the cylinder below a given disk slice, so that the boundary of this "disk" is a constant $S^{n-1} \times \{0\}$. This will then give a homotopy between the given map and a map that maps all of $\mathbb{E}^n$ into A, since $f[\mathbb{E}^n \times \{1\} \cup S^{n-1} \times I]$ has the first component to be x_0 and the $S^{n-1} \times I$ is all in A since S^{n-1} is mapped into A by definition of relative homotopy. Thus we have a homotopy between f and something contained in A, as desired.

Theorem 3. *The sequence given by*

$$\cdots \to \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \xrightarrow{i_*} \pi_{n-1}(X, x_0) \to \cdots$$

Where the maps j_* and i_* are inclusions, and the map ∂ is the restriction to I^{n-1}. Is a long exact sequence.

First, let's think about j_* a bit more concretely, if we have a map $(S^n, 1) \to (A, x_0)$ this is also a map $(S^n, 1) \to (X, x_0)$. Similarly, since $x_0 \in A$ mapping all of ∂I^n to x_0 certainly counts as mapping all of it into A, so i_* is an inclusion as well. As for ∂, this map is given by restricting maps to the face I^{n-1}, by the way we've defined J^{n-1} it turns out that $J^{n-1} \cap I^{n-1} = \partial I^{n-1}$, so that this is also nicely behaved.

Proof. Following Hatcher (4.3), we prove this in a more general setting. We look instead at the case where we have a second subspace $B \subset A$ and then show that

$$\cdots \to \pi_n(X, B, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \xrightarrow{i_*} \pi_{n-1}(X, B, x_0) \to \cdots$$

is exact. Taking $B = x_0$ proves the theorem. I will break this down into three steps, to demonstrate exactness at each individual term, I will then break this down further into two steps, one for each inclusion.

- $\pi_n(X, B, x_0)$

- the composition $J_* i_*$ is 0 because it (is homotopic to something that) maps all of $\mathbb{E}^n$ into A, since that is what it means to be in $\pi_n(A, B, x_0)$.
- Let $f \in \ker(j_*)$ then f is homotopic (relative to B) to something that maps into A, which places it in the image of i_*

- $\pi_n(X, A, x_0)$
  - Since the element comes from $\pi_n(X, B, x_0)$ the boundary of I^n lies entirely in B, so that once we take the boundary map we find that it is zero in $\pi_n(A, B, x_0)$, since all of it is in B.
  - Now say that $f \in \ker(\partial)$ then the restriction of f to I^{n-1} is homotopic to something contained in B. But this homotopy is then $F : I^{n-1} \times I \to A$ with one side agreeing with a side of f, so we can attach these two maps. This new map is homotopic to f by the homotopy that slowly builds F. But this map is in the image of j_*, since it has B for its edge.

- $\pi_n(A, B, x_0)$
  - The image $\overline{f}$ of a map f originating in $\pi_{n+1}(X, A, x_0)$ under the boundary map is homotopic to a constant map via the original map f, except that possibly we need the ambient space X to move through, but passing to $\pi_n(X, B, x_0)$ provides exactly that.
  - On the other hand, given a $f \in \pi_n(A, B, x_0)$ homotopic via F to a constant map inside $\pi_n(X, B, x_0)$ we can use F to construct an element of $\pi_{n+1}(X, A, x_0)$ with image f under the boundary map. We do this by taking $g = F|_{I^n \times I}$ i.e. the side of F that roams over B. We then shift g homotopically onto the same side as f so that we have that f with g added is in the image of the boundary map. However, adding another map is something that can be done via homotopy, so that f itself is also in the image of the boundary map.

■

Now that we've demonstrated the existence of that sequence, let's get to applying it. But first we'll need another piece of the puzzle.

4.3 The inclusion isomorphism

One last thing we'll need is a theorem about higher homotopy groups of cell complexes.

Theorem 4. *For a cell complex X that can be written as a union of cell complexes A and B with nonempty connected intersection C. Then if $\pi_j(A, C) = \pi_j(B, C) = 0$ for $j = 0, 1, \ldots, n$ then the inclusion map $\pi_i(A, C) \to \pi_i(X, B)$ is an isomorphism for $i < 2n$.*

(We omit the proof of this theorem, it is theorem 4.23 in Hatcher)

4.4 Stability

We are now ready to introduce stability, which is central to homotopy theory, beyond its application to the main diagonal $\pi_n(S^n)$. We will be showing that $\pi_n(S^n) \cong \pi_{n+1}(S^{n+1})$, we will do this in a 3 step process, via the cone and suspension. Altogether we are showing that $\pi_n(S^n) \cong \pi_{n+1}(CS^n, S^n) \cong \pi_{n+1}(\Sigma S^n, CS^n) \cong \pi_{n+1}(S^{n+1})$.

We take in the above long exact sequence, for a connected space Y, a specific example of a superspace X, so that the sequence will yield a set of isomorphisms. We take $X = CY$, the cone of Y, and $A = Y$. We then get that $\pi_i(X) = \pi_i(CY) = 0$ since a cone of a connected space is retractable. The exact sequence then gives $0 \to \pi_n(CY, Y) \xrightarrow{\partial} \pi_{n-1}(Y) \to 0$ so that $\pi_n(CY, Y) \cong \pi_{n-1}(Y)$. This gives us our first desired isomorphism, namely that $\pi_n(S^n) \cong \pi_{n+1}(CS^n, S^n)$. Note, however, the generality of the result.

For the second isomorphism we use the inclusion isomorphism stated above in (4.3). Taking there $X = S^{n+1} = \Sigma S^n$, for as you may recall, then suspension of S^n is in fact S^{n+1}. We then take $A = \mathbb{E}^{n+1}_+ \cong C_+ S^n$ the top hemisphere of S^{n+1}, and similarly $B = \mathbb{E}^{n+1}_- \cong C_- S^n$, the bottom hemisphere of S^{n+1}. We can see that in fact $\mathbb{E}^{n+1}_+ \cong C_+ S^n$ by imagining the terminal point of the cone to be the center of S^n and then the cone consists of the map shrinking the sphere to a point. We then get that $C = A \cap B = S^n$. Then by the inclusion we get the isomorphism $\pi_i(CS^n, S^n) \cong \pi_i(\mathbb{E}^{n+1}, S^n) \cong \pi_i(S^{n+1}, \mathbb{E}^{n+1}_-) \cong \pi_i(\Sigma S^n, CS^n)$ for $i < 2n-1$. This holds here because we have that $\pi_j(\mathbb{E}^{n+1}, S^n) = 0$ for $j = 0, \ldots, n-1$ (this follows from $\pi_j(S^n) = 0$, which you will recall we showed in section 3 using cellular approximation). So that we get an isomorphism when $i < 2(n-1) \Leftrightarrow i+1 < 2n-1$.

Finally, we show that $\pi_{n+1}(\Sigma S^n, CS^n) \cong \pi_{n+1}(S^{n+1})$. We do this again by using the long exact sequence of relative homotopy. Here we take, for a given space Y, our ambient space to be $X = \Sigma Y$, the suspension or double cone of Y, and the the subspace $A = CY \subset X$, the cone of Y. We have that $\pi_i(A, x_0) = \pi_i(CY, x_0) = 0$ so that we then get $0 \to \pi_n(\Sigma Y) \to \pi_n(\Sigma Y, CY) \to 0$ which gives us the isomorphism $\pi_n(\Sigma Y) \cong \pi_n(\Sigma Y, CY)$. But, as mentioned above, when we take $Y = S^n$ then we get that the suspension ΣY is equal to $\Sigma S^n = S^{n+1}$. So that by combining these three isomorphisms, we then get the isomorphism $\pi_n(S^n) \cong \pi_{n+1}(CS^n, S^n) \cong \pi_{n+1}(\Sigma S^n, CS^n) \cong \pi_{n+1}(\Sigma S^n) = \pi_{n+1}(S^{n+1})$.

This is the desired isomorphism, although it breaks down when we get to π_1, because we had the condition that $i+1 < 2n-1$ and we are taking $i = n$, so that this holds when $n > 2$, ,but note that we need this isomorphism for $n+1$, so in fact this holds exactly when $n > 1$. Thus we will still need to use other methods to show that $\pi_1(S^1) \cong \pi_2(S^2)$. However, once that has been shown, the above yields that $\pi_n(S^n) = \mathbb{Z}$ for all n.

4.5 The general case

Note that the above didn't actually really use the fact that the degree of the homotopy group and sphere were identical. There is in fact a neat and very useful generalization of the above result. Although in general it turns out that homotopy groups of spheres are rather complicated, and can be any of a wide variety of (mostly finite abelian) groups, eventually, simultaneously increasing the degree of the group and of the sphere by 1 produces the same group. So that we in fact have stability across a wide range of diagonals. I.e. we have, for sufficiently large k, that $\pi_k(S^{n+k})$ is the same for all k.

In particular, given some $\pi_a(S^b)$, we can use the same exact sequence as above to get that the sequence stabilizes i.e. $\pi_{a+k}(S^{b+k})$ is independent of k as soon as we take k to be sufficiently large. So that if we start with any $\pi_a(S^b)$ and move to $\pi_{a+k}(S^{b+k})$ for increasing k, we will soon keep getting the same answer, and this is the stability of Homotopy groups. We denote these as π_n^s, and call such a group the n-th stable homotopy group.

5 The Hopf map

We now pass to a central piece of the higher homotopy puzzle. This is a quite exotic map from S^3 to S^2, called the Hopf map. It will give us a handful of interesting facts about homotopy. But first, some more motivation.

5.1 Hints of homology

When mathematicians first started working with higher homotopy, they noticed a striking fact, that it was fairly similar to homology. It turns out, that for homology, we have that $H_n(S^n) = \mathbb{Z}$ and $H_n(S^m) = 0$ for $m \neq n$. As you can see, all the results we've shown thus far have been consistent with this pattern. So all that was needed to show was that $\pi_n(S^m) = 0$ even when $m > n$, and we would be well on our way to showing the equivalence of homology and higher homotopy, at least their agreement on S^n would be demonstrated. Then came along Hopf fibration, and its associated long exact sequence, and messed this whole equivalence up by computing $\pi_3(S^2)$.

5.2 Fibration and its long exact sequence

A *fibration* is a pair of spaces E, B and a map $p : E \to B$ such that for any homotopy $f_t : X \to B$, and a partial lift $p \circ \widetilde{f}_0 = f_0$, we can extend this to a full lift $\widetilde{f}_t$. If this is reminiscent of covering spaces, that's because every covering space is a fibration. More specifically, fibration is the appropriate generalization of covering space to higher dimensional homotopy. A trivial example of a fibration is a map $p : B \times F \to B$, for any pair of spaces F, B. Since we can lift a homotopy to be constant on the second entry.

Hopf introduced a specific very interesting fibration, which maps S^3 to S^2 with kernel S^1. But first, we need to show why fibrations are interesting in the first place.

For a fibration, we generally write $F \to E \xrightarrow{p} B$, where F is the kernel of p (i.e. the pre-image of a specified point $b_0 \in B$), so that this sequence is exact at E (this generally needn't be the case, and in the case of exactness we call this a fiber bundle, but we do not make these distinctions here). We now obtain a long exact sequence of homotopy groups, a special case of the more general Puppe sequence. The sequence is

$$\cdots \to \pi_{n+1}(E) \to \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

The map $\pi_{n+1}(B) \to \pi_n(F)$ is somewhat subtle, and is essentially constructed using some diagram manipulation. We start by pulling back a map $\phi : S^{n+1} \to B$ to a map $\theta : \mathbb{E}^{n+1} \to E$ by lifting the homotopy. We then pull this back further to a map $S^n \to F$ using the fact that the inclusion map $S^n \to \mathbb{E}^{n+1}$ composed with the map $\mathbb{E}^{n+1} \to S^{n+1}$ (given by collapsing the boundary) has the image of S^n just a point. So this tells us that if we then composed that with ϕ this is equivalent to composing the inclusion map $S^n \to \mathbb{E}^{n+1}$ with the composition of p and θ, and since the image here is the basepoint, by exactness this originates in F, so we then get a map $\psi : S^n \to F$. We set this ψ to be the image of ϕ. Confirmation of exactness can also be done by applying the homotopy lifting repeatedly, we omit this because it is tiring.

5.3 The projective bundle

We in general, for any n, get a fiber bundle $S^1 \to S^{2n+1} \to \mathbb{C}P^n$. We are constructing this by viewing S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$. Then we note than $\mathbb{C}P^n$ is the quotient of $\mathbb{C}^{n+1}/\sim$ by complex lines $z \sim \lambda z$. We can, as in the real case, get all of projective space just from the unit sphere. In the real case, the inverse image of a point will be a pair of antipodal points. In the complex case, the preimage of z is λz with $|\lambda z| = 1$. Given any point on the complex sphere, multiplying by λ with $|\lambda| = 1$ i.e. $\lambda \in S^1 \subset \mathbb{C}$ preserves the norm. Therefore, the preimage of a point under the map $S^{2n+1} \to \mathbb{C}P^n$ is a circle S^1. This yields $S^1 \to S^{2n+1} \xrightarrow{p} \mathbb{C}P^n$, where the map p is $(z_0, \ldots, z_n) \mapsto [z_0, \ldots, z_n]$.

5.4 Hopf bundle

We can then in the above take $n = 1$. It turns out that $\mathbb{C}P^1 \cong S^2$. We can see this by noticing they have the same cell complex construction, a disk attached to a point, which makes sense from the perspective that $\mathbb{C}P^1$ is $\mathbb{C}$ with a compactification point at infinity. This works because points of $\mathbb{C}P^1$ are pairs of complex numbers up to scalers, so that they look like z_1/z_2, which is $\mathbb{C}$ (take $z_2 = 1$) along with a point at infinity (for $z_2 = 0$). In fact, this is the projection $S^3 \xrightarrow{p} S^2$, it is given by the quotient of two complex numbers on the sphere.

6 Equivalence induced by Hopf

Using the Hopf fibration, we get the long exact sequence

$$\cdots \to \pi_n(S^1) \to \pi_n(S^3) \to \pi_n(S^2) \to \pi_{n-1}(S^1) \to \cdots$$

One immediate result is that we get

$$0 = \pi_2(S^3) \to \pi_2(S^2) \to \pi_1(S^1) \to \pi_1(S^3) = 0$$

so that $\pi_2(S^2) \cong \pi_1(S^1) \cong \mathbb{Z}$, a result we needed earlier. But the other thing it shows is that for $n \geq 3$ i.e. when $n - 1 \geq 2$, so that we then have $\pi_{n-1}(S^1) = 0$, we then get:

$$0 = \pi_n(S^1) \to \pi_n(S^3) \to \pi_n(S^2) \to \pi_{n-1}(S^1) = 0$$

So that we get $\pi_n(S^3) \cong \pi_n(S^2)$ for all $n \geq 3$, in particular we get that $\pi_3(S^2) \cong \mathbb{Z}$, which is the first place in which homotopy differs from homology.

7 The general case

There are also two more Hopf maps, given by the quaternians and octonians. These give us fiber bundles $S^3 \to S^7 \to S^4$ and $S^7 \to S^{15} \to S^8$. In order to find the higher homotopy groups of a sphere, we use methods similar to those used here for the easier cases. Namely, using these Hopf maps, suspensions, long exact sequences, and other diagrams. As you may have gotten a sense in this exposition, these are generally fairly difficult to compute, and cleverness must be employed.

Sources:

Hatcher, Allen. 2001. *Algebraic Topology.* New York, Cambridge University Press.